ALGEBRAIC ISOMORPHISM IN TWO-DIMENSIONAL ANOMALOUS GAUGE THEORIES

C. G. Carvalhaes[1,3], L. V. Belvedere[1], H. Boschi Filho[2],

C. P. Natividade[1]

1*Instituto de Física - Universidade Federal Fluminense*

Av. Litorânea, S/N, Boa Viagem, Niterói, CEP.24210-340, Rio de Janeiro - Brasil

2*Instituto de Física - Universidade Federal do Rio de Janeiro*

Ilha do Fundão, CEP.21945, Rio de Janeiro, Brasil

3*Instituto de Matemática e Estatística - Universidade do Estado do Rio de Janeiro*

Rua São Francisco Xavier, 524, Maracanã, CEP. 20559 900, Rio de Janeiro, Brasil

Abstract

The operator solution of the anomalous chiral Schwinger model is discussed on the basis of the general principles of Wightman field theory. Some basic structural properties of the model are analyzed taking a careful control on the Hilbert space associated with the Wightman functions. The isomorphism between gauge noninvariant and gauge invariant descriptions of the anomalous theory is established in terms of the corresponding field algebras. We show that (*i*) the Θ-vacuum representation and (*ii*) the suggested equivalence of vector Schwinger model and chiral Schwinger model cannot be established in terms of the intrinsic field algebra.

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1

Isomorphism in Anomalous Theories

Luiz Victorio Belvedere

Instituto de Física

Universidade Federal Fluminense

CEP: 24210-340 Niterói - RJ - Brazil

Fax: 55 021 620-6735

E-mail: belve@if.uff.br

I. INTRODUCTION

The chiral Schwinger model (CSM) [1] has been intensively studied [2] since it provides a very instructive theoretical laboratory for studying the possibility of constructing a consistent quantum gauge theory exhibiting anomalous breaking of a local gauge symmetry.

Following the proposal of Faddeev and Satashvili [3], in a quantum theory where gauge invariance is spoiled by anomalies, gauge invariance can be restored with the introduction of *ad hoc* extra degrees of freedom into the theory by adding to the original Lagrangian a Wess-Zumino (WZ) term [4].

However, several authors [5-6] realized that the WZ term does not need to be introduced *ad hoc* in order to naturally embed an anomalous theory into a gauge theory. Using the path integral formalism and the Faddeev-Popov prescription, the WZ action emerges as being related to the Jacobian of the change in the fermionic measure due to a chiral gauge transformation. From the operator point of view the gauge invariant (GI) formulation is obtained by performing an operator-valued gauge transformation on the gauge noninvariant (GNI) formulation [7].

Using the path-integral formalism and the Hamiltonian formulation, it was shown in Ref.[6] that *anomalous chiral gauge theories exhibit a peculiar feature which allows two isomorphic formulations: the GNI and the GI formulations.* From the path-integral point of view, the isomorphism is established between the Wightman functions that are representations of the local field algebra generated from the intrinsic irreducible set of field operators defining the GNI formulation, and the corresponding Wightman functions of the gauge transformed field operators defining the GI formulation of the model. Using the Hamiltonian formalism it was shown that for the GI formulation of the CSM one can construct a set of gauge invariant operators whose equal-time commutator algebra is isomorphic to the equal-time commutator algebra of the phase space variables of the GNI formulation of the model. This implies that the constraints and the canonical Hamiltonian of GI description map into the corresponding quantities of the GNI one. As was stressed in Ref. [6], *the*

isomorphism between the two formulations means that the GNI formulation describes the
gauge invariant sector of the enlarged GI formulation.

However, in spite of the existence of a sizable number of works on this subject, some questions related with basic structural properties of the anomalous chiral model have not been fully appreciated and clarified in the preceding literature, such as : (i) the implementation at the operator level of the isomorphism between the GNI and GI formulations of the CSM, as proposed in Ref.[6] in the path-integral and Hamiltonian formalisms; (ii) the need for a Θ-vacuum parametrization in the model, as claimed in Ref.[9] and (iii) the equivalence of the vector Schwinger model (VSM) and the GI formulation of CSM defined for the regularization dependent parameter $a = 2$ as suggested in Refs.[9,10]. It is the purpose of this paper to discuss the fine mathematical aspects of these questions, as well as, to acquire a more detailed understanding of the physical grounds of the gauge invariance implied by the addition of the WZ piece into the anomalous theory.

We shall discuss the operator solution of the anomalous CSM on the basis of the general principles of Wightman field theory. The guideline of our approach of the anomalous theory will be inspired in the rigorous treatment given in Ref.[11] for the VSM. In this approach we only use the field algebra generated from the intrinsic irreducible set of field operators of the theory [16] in order to exert a careful control on the Hilbert space associated with the Wightman functions that define the theory. As we shall see, by relaxing this rigorous control on the construction of the Hilbert space of the anomalous theory some misleading conclusions can arise, as for example the existence of an infinite number of states which are degenerated in energy with the vacuum state [9] and the equivalence of CSM and VSM, as proposed in Refs. [9,10]. In order to make clear that the above conclusions rest on the incorrect choice of the observables representing the fermionic content of the model, we shall adopt the same bosonization scheme employed in Ref.[9].

Using the standard Mandelstam bosonization scheme the CSM is considered in Ref.[12]. The operator solution used introduces a minimal Bose field algebra (*minimal bosonization scheme*) such that no spurious degrees of freedom appears, and the Maxwell equation dis-

4

playing the dynamical breakdown of gauge invariance holds in the strong form and is satisfied as an operator identity.

In Refs.[8,9] the vector and chiral Schwinger models are considered through a quantization procedure that takes into account the fact that the corresponding bosonized Lagrangian densities contain higher derivative terms and derivative interactions. This bosonization scheme for higher-derivative field theories (*enlarged bosonization scheme*) introduces degrees of freedom quantized with negative metric that generate spurious operators. This approach resembles the local gauge formulation of quantum field theory in which we have locality at the expense of weakening the local Gauss' law. The Gauss' law is modified by the introduction of a longitudinal current and the Hilbert space realization of the theory contains unphysical states. In the vector case the solution obtained in Refs.[8,9] coincides with the Lowenstein-Swieca solution [13]. Nevertheless, due to the absence of gauge invariance, in the anomalous CSM the additional gauge degree of freedom is not eliminated and the solution exhibits an enlarged Bose field algebra that contains redundant spurious operators. In spite of the existence of redundant degrees of freedom in the solution given in Refs.[8,9], by taking into account the Hilbert space that is the representation of the Wightman functions defining the model, we show that the physical content of the solutions given in Refs.[8,9] and by the minimal bosonization scheme used in Ref.[12] is the same, since they are related by a superfluous spurious phase operator. Since in the CSM the cluster decomposition property is not violated for Wightman functions that are representations of the intrinsic field algebra that defines the model[1], the Θ-vacuum parametrization suggested by the authors of Ref.[9] becomes completely unnecessary. Of course, the suggested equivalence of the CSM defined for $a = 2$ and the VSM cannot be established if we consider the Hilbert space in which the intrinsic field algebra of the model is represented. The equivalence proposed in Ref.[9] is a consequence of an incorrect decomposition of the closure of the Hilbert space that implies

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[1] *In agreement with the results of Refs.[8,12].*

the choice of a field operator that does not belong to the intrinsic local field algebra to represent the fermionic content of the model. Within the approach based on the intrinsic field algebra, the existence of Θ-vacuum and the equivalence with the VSM suggested in Refs.[9,10] cannot be regarded as being structural properties of the CSM since they are dependent on the use of a redundant algebra, rather than on the field algebra which defines the model.

The paper is organized as follows. In section 2 the GNI formulation of the model is discussed and we show that the cluster decomposition property is not violated for the Wightman functions that are representations of the field algebra generated from the irreducible set of field operators defining the model. The case $a = 2$ is considered and we show that the factorization of the completion of states performed in Refs.[9,10] leads to some improper conclusions about basic structural properties of the model. In section 3 we consider the GI formulation of the model and the isomorphism between the intrinsic operator field algebra defining the GI formulation and the intrinsic field algebra defining the GNI formulation is established. The role played by the WZ field in the implementability of extended local gauge transformations is also discussed.

II. GAUGE NONINVARIANT FORMULATION

The CSM is a two-dimensional field theory defined from the classical Jackiw-Rajaraman (JR) Lagrangian density [2] [1]

$$\mathcal{L}_{JR} = -\frac{1}{4}\left(\mathcal{F}_{\mu\nu}\right)^2 + i\,\bar{\psi}\,\partial_\mu\,\gamma^\mu\,\psi - g\,\mathcal{A}_\mu\,\bar{\psi}\,\gamma^\mu\,P_+\,\psi \;. \tag{2.1}$$

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[2] *The conventions used are:* $\psi = (\psi_\ell\,,\,\psi_r)^T$, $\epsilon^{01} = g^{00} = -g^{11} = 1$, $\tilde{\partial}^\mu \equiv \epsilon^{\mu\nu}\,\partial_\nu$,

$$\gamma^5 = -\gamma^0\gamma^1\,,\;\gamma^\mu\gamma^5 = \epsilon^{\mu\nu}\,\gamma_\nu\;,$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,,\;\gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,,\;\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\;.$$

Here $\mathcal{F}_{\mu\nu}$ denotes, as usual, the field-strength tensor and $P_+ = (1 + \gamma^5)$ projects out the left-moving fermion. The classical Lagrangian density (2.1) exhibits invariance under the local gauge transformation

$$\psi \longrightarrow e^{i P_+ \Lambda(x)} \psi , \qquad (2.2a)$$

$$\mathcal{A}_\mu \longrightarrow \mathcal{A}_\mu + \frac{1}{g} \partial_\mu \Lambda(x) . \qquad (2.2b)$$

At the quantum level the chiral anomaly spoils the gauge invariance and the model is described by the gauge noninvariant effective Lagrangian [7],

$$\mathcal{L}_{GNI} = -\frac{1}{4} (\mathcal{F}_{\mu\nu})^2 + i \bar{\psi} \partial_\mu \gamma^\mu \psi + \frac{g}{\sqrt{\pi}} \mathcal{K}_\ell^\mu \mathcal{A}_\mu + \frac{ag^2}{2\pi} (\mathcal{A}_\mu)^2 , \qquad (2.3)$$

where a is the JR parameter that characterizes the ambiguity in the quantization of the model and $\mathcal{K}_\ell^\mu$ is the left-current defined by the point-splitting regularization, independent of the JR parameter [7]

$$\mathcal{K}_\ell^\mu = 2 \sqrt{\pi} \lim_{\varepsilon \to 0} \left[\bar{\psi}(x+\varepsilon) \gamma^\mu P_+ \psi(x) - V.E.V. \right] \left[1 + ie\sqrt{\pi} P_+ (g_{\mu\nu} - \epsilon_{\mu\nu}) \varepsilon^\mu \mathcal{A}^\nu(x) \right] . \quad (2.4)$$

The bosonized effective GNI Lagrangian density is [1]

$$\mathcal{L}_{GNI} = -\frac{1}{4} (\mathcal{F}_{\mu\nu})^2 + \frac{1}{2} (\partial^\mu \phi)^2 + \frac{g}{\sqrt{\pi}} (\partial^\mu \phi + \tilde{\partial}^\mu \phi) \mathcal{A}_\mu + \frac{ag^2}{2\pi} (\mathcal{A}_\mu)^2 , \qquad (2.5)$$

and the resulting equations of motion are

$$\Box \phi + \frac{g}{\sqrt{\pi}} (\partial^\nu + \tilde{\partial}^\nu) \mathcal{A}_\nu = 0 , \qquad (2.6a)$$

$$\partial_\mu \mathcal{F}^{\mu\nu} = J^\nu = -\frac{g}{\sqrt{\pi}} (\partial^\nu + \tilde{\partial}^\nu) \phi - a \frac{g^2}{\pi} \mathcal{A}^\nu . \qquad (2.6b)$$

Introducing a local gauge decomposition for the gauge field

$$\mathcal{A}^\mu = \tilde{\partial}^\mu \chi + \partial^\mu \lambda , \qquad (2.7)$$

the field-strength tensor is given by

$$\mathcal{F}^{\mu\nu} = -\epsilon^{\mu\nu} \Box \chi . \qquad (2.8)$$

The Maxwell Lagrangian can be written as

$$\mathcal{L}_M = -\frac{1}{4}(\mathcal{F}^{\mu\nu})^2 = \frac{1}{2}\Box\chi\Box\chi \,, \tag{2.9}$$

and the bosonized Lagrangian density (2.5) can be treated as a higher-derivative field theory [8,9,14].

For our purposes it is therefore convenient to start with the *enlarged bosonization scheme* used in Refs.[8,9,14] and introduce the field transformations $(a \neq 1)$ [3]

$$\phi = \phi' + \frac{g}{\sqrt{\pi}}\left(\chi - \lambda\right) \,, \tag{2.10a}$$

$$\chi_1 = \frac{1}{m_a}\Box\chi \,, \tag{2.10b}$$

$$\chi_2 = \frac{1}{m_a}(\Box + m_a^2)\chi \,, \tag{2.10c}$$

$$\lambda = \lambda' - \frac{1}{a-1}\chi \,, \tag{2.10d}$$

with $m_a = g\,a\,/\sqrt{\pi\,(a-1)}$. The Lagrangian density (2.5) of the GNI formulation is then reduced to a local one [4] [8,9]

$$\mathcal{L}_{GNI} = \frac{1}{2}(\partial_\mu\phi')^2 - \frac{1}{2}(\partial_\mu\chi_2)^2 + \frac{g^2}{2\pi}(a-1)(\partial_\mu\lambda')^2 + \frac{1}{2}(\partial_\mu\chi_1)^2 - \frac{1}{2}m_a^2\chi_1^2 \,. \tag{2.11}$$

The field χ_2 is a free and massless field quantized with negative metric, and λ' is a non-canonical free massless field. In the VSM the gauge invariance ensures that the field λ' is a pure gauge excitation and does not appears in the bosonized Lagrangian density [8,14]. However, *in the anomalous chiral model the additional degree of freedom λ' is a dynamical field and, as we shall see, its presence ensures the existence of fermions in the asymptotic states*, implying that the screening and confinement aspects exhibited by the CSM differ

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[3] *For the benefit of the reader we shall use the same notation adopted in Refs.[8,9,14].*

[4] *As stressed in Ref.[8], these higher-derivative field theory transformations are consistent with the Dirac brackets of the constrained theory obtained in Refs.[18].*

from those of the VSM [5]. This behavior occurs for all values of the regularization dependent parameter $a > 1$.

Let us consider the equation of motion (2.6b). In terms of the new fields (2.10), we can rewrite the vector current as

$$J^\mu = m_a \, \tilde{\partial}^\mu \chi_1 + L^\mu \,, \tag{2.12}$$

where L^μ is a longitudinal current ($\Box \, L^\mu = 0$) given in terms of derivatives of the massless fields by

$$L^\mu = -\frac{g}{\sqrt{\pi}} \left\{ (\partial^\mu + \tilde{\partial}^\mu) \phi' + \frac{a}{\sqrt{a-1}} \tilde{\partial}^\mu \chi_2 + \frac{g}{\sqrt{\pi}} \left[(a-1)\partial^\mu \lambda' - \tilde{\partial}^\mu \lambda' \right] \right\} \,. \tag{2.13}$$

The first term in Eq.(2.13) represents the free left-moving Fermion current j_ℓ^μ. The negative metric quantization of the field χ_2 ensures that

$$[\, L^\mu(x) , \, L^\nu(y) \,] = 0 \ , \forall \, x, y \,, \tag{2.14}$$

and L^μ generates, from the vacuum, zero norm states

$$\langle \, L_\mu^* L_\nu \, \rangle_o \equiv \langle \, L_\mu \Psi_o , \, L_\nu \Psi_o \, \rangle = 0 \,, \tag{2.15}$$

where Ψ_o is the vacuum vector.

For further convenience we introduce the dual field [8,9]

$$\tilde{\phi} = \tilde{\phi}' - \frac{g}{\sqrt{\pi}} (\chi - \lambda) \,, \tag{2.16}$$

which enables the enlargement of the algebra of the Bose fields by the introduction of two independent right and left mover fields

$$\phi_r = \frac{1}{2}(\phi + \tilde{\phi}) = \phi_r' \,, \tag{2.17a}$$

$$\phi_\ell = \frac{1}{2}(\phi - \tilde{\phi}) = \phi_\ell' + \frac{1}{\sqrt{a-1}} (\chi_2 - \chi_1) - \frac{g}{\sqrt{\pi}} \lambda' \,, \tag{2.17b}$$

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[5] *As we shall see the non-trivial anomalous nature of the field λ' ensures the dynamics for the WZ field.*

9

$$L_r = \frac{1}{2}(L + \tilde{L}) = \frac{g}{\sqrt{\pi}} a \lambda'_r - \frac{a}{\sqrt{a-1}} \chi_{2r} , \qquad (2.17c)$$

$$L_\ell = \frac{1}{2}(L - \tilde{L}) = 2 \phi'_\ell + \frac{g}{\sqrt{\pi}}(a-2) \lambda'_\ell + \frac{a}{\sqrt{a-1}} \chi_{2\ell} , \qquad (2.17d)$$

where L is the potential for the longitudinal current (2.13)

$$L^\mu = -\frac{g}{\sqrt{\pi}} \partial^\mu L = \varepsilon^{\mu\nu} L_\mu^5 = \frac{g}{\sqrt{\pi}} \tilde{\partial}^\mu \tilde{L} . \qquad (2.18)$$

The operator solution of the equations of motion is constructed from the set of Bose fields $\{ \chi_1, \chi_2, \phi', \lambda' \}$, the so-called "*building blocks*" [11], and is given by

$$\mathcal{A}^\mu(x) = \frac{\sqrt{\pi}}{ga\sqrt{a-1}} \left\{ (a-1)\tilde{\partial}^\mu \left(\chi_2(x) - \chi_1(x) \right) - \partial^\mu \left(\chi_2(x) - \chi_1(x) \right) \right\} + \partial^\mu \lambda'(x) , \quad (2.19a)$$

$$\psi_r^0(x) = \left(\frac{\mu_o}{2\pi}\right)^{1/2} : e^{2i\sqrt{\pi}\phi_r(x)} : = \left(\frac{\mu_o}{2\pi}\right)^{1/2} : e^{2i\sqrt{\pi}\phi'_r(x)} : , \qquad (2.19b)$$

$$\psi_\ell(x) = \left(\frac{\mu_o}{2\pi}\right)^{1/2} : e^{2i\sqrt{\pi}\phi_\ell(x)} : =: e^{-2i\sqrt{\pi/(a-1)}\chi_1}(x) : \psi_\ell^0(x)\,\Gamma(x) , \qquad (2.19c)$$

where μ_o is an arbitrary finite mass scale, $\psi_\ell^0(x)$ is the free left-moving Fermi field given by

$$\psi_\ell^0(x) = \left(\frac{\mu_o}{2\pi}\right)^{1/2} : e^{2i\sqrt{\pi}\phi'_\ell(x)} : , \qquad (2.20)$$

and

$$\Gamma(x) =: e^{2i\sqrt{\pi}\left(\frac{1}{\sqrt{a-1}}\chi_2(x) - \frac{g}{\sqrt{\pi}}\lambda'(x)\right)} : . \qquad (2.21)$$

The Wick exponential $: \exp i\Phi :$ has to be understood as a formal series of Wick-ordered powers of the field Φ at the exponent[6] [17].

The so-obtained fermionic fields (2.19b-c) satisfy abnormal commutation relations. The Fermi field operators with correct anticommutation relations can be obtained from the original set by a Klein transformation [16]. In what follows we shall suppress the Klein factors, since they are not needed for our present purposes.

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[6] *The problem for defining the Wick exponential of massless scalar fields in two dimensions is that the corresponding Wightman functions do not satisfy positivity unless some selection rule is introduced [19]. In Ref.[17] the definition of the Wick exponential of the massless scalar field in two dimensions as an operator-valued distribution is discussed in the Krein space realization of the field.*

A. Intrinsic Local Field Algebra

The procedure which we shall adopt to display the basic structural properties of the model and obtain a consistent prescription to identify correctly its physical content, is to embed the mathematical structures of the bosonization scheme into the context of the general principles of Wightman field theory. In this way the states that are admissible on the Hilbert space $\mathcal{H}$ are selected according to some appropriate criteria and the physical interpretation of the theory is based on the Wightman functions representing the intrinsic local field algebra which describes the degrees of freedom of the model.

Within a precise mathematical point of view, a relativistic Quantum Field Theory is formulated in terms of fields as local operator-valued tempered distributions. Due to the singular character of the fields at a point, only space-time averages of fields have physical meaning, in general. In this way, in a Quantum Field Theory the role of the algebra of the canonical variables is played by the *field algebra* generated by polynomials of the smeared fields $\Phi_i(f)$,

$$\Phi_i(f) \,=\, \int \Phi_i(x)\, f(x)\, d^n x$$

with $f(x)$ a smooth and fast decreasing regular test function.

A *local field algebra* $\Im$ satisfies the algebraic constraint given by locality,

$$[\,\Phi(f)\,,\,\Phi(h)\,] \,=\, 0$$

if supp f is space-like relative to supp h. This property expresses the fact that fields averages belonging to space-like separated regions are simultaneously diagonalizable operators. Therefore, a *local Quantum Field Theory* is specified by a representation of the algebra $\Im$ generated by the polynomials of the smoothed local fields $\Phi(f)$ satisfying the locality condition.

Given a QFT with a set of local fields $\{\Phi\}$. The set of fields $\{\Phi\}$ provides a complete

description of the system if for every space-like slab $\mathcal{S}$ [7], the set of vectors $\wp\,(\Phi(f))\Psi_o$ spans the Hilbert space $\mathcal{H}$. Here $\wp$ is an arbitrary polynomial in the $\Phi(f)$, f having supports in $\mathcal{S}$. Equivalently, the set of fields $\{\Phi\}$ provides a complete description of the system if for every space-like slab $\mathcal{S}$, the sets $\{\Phi(f)\}$, f with supp. in $\mathcal{S}$, are irreducible [8].

The operator solution of the equations of motion defining the GNI formulation of the CSM is given in terms of the *irreducible set of field operators* $\{\bar{\psi},\psi,\mathcal{A}_\mu\}$ [11,16]. These field operators constitute the intrinsic mathematical description of the model and serve as a kind of building material in terms of which the GNI version of the model is formulated and whose Wightman functions define the model. Every operator of the theory is a function of the intrinsic set of field operators $\{\bar{\psi},\psi,\mathcal{A}_\mu\}$ which defines a polynomial algebra $\Im_{GNI}$, that is, *the local field algebra $\Im_{GNI}$ intrinsic to the GNI formulation is generated from the irreducible set of field operators $\{\bar{\psi},\psi,\mathcal{A}_\mu\}$, through polynomials of these smeared fields,* Wick ordering, point-splitting regularization of polynomials, etc. The Wightman functions generated from the intrinsic field algebra $\Im_{GNI}$ identifies a vector space $\mathcal{D}_o \equiv \Im_{GNI}\,\Psi_o$ of local states: the field algebra $\Im_{GNI}$ is represented in the Hilbert space $\mathcal{H}_{GNI} \doteq \Im_{GNI}\,\Psi_o$ of the GNI formulation.

Within the bosonization scheme, the operator solution is constructed in terms of Wick exponentials and derivatives of scalar fields in two dimensions. The introduction of the set

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[7] *A space-like slab in space-time is defined as the open set between two parallel space-like planes.*

[8] *The set of fields $\{\Phi(f)\}$ is irreducible if every bounded operator which has the properties,*

$$\mathcal{O}\mathcal{D} \subset \mathcal{D}\,,$$

where $\mathcal{D}$ is the common dense domain of the set $\{\Phi(f)\}$, and

$$\Phi(f)\,\mathcal{O} \,=\, \mathcal{O}\,\Phi(f)\,,$$

is a constant multiple of the identity.

of Bose fields $\{\phi', \chi_2, \lambda', \chi_1\}$ defines an enlarged redundant field algebra $\mathfrak{F}^B_{GNI}$ which is represented on the Hilbert space $\mathcal{H}^B_{GNI} \doteq \mathfrak{F}^B_{GNI} \Psi_o$, which provides a Fock-Krein representation[9] of the algebra $\mathfrak{F}^B_{GNI}$ [11]. The Bose fields $\{\phi', \chi_2, \lambda', \chi_1\}$ are the *building blocks* in terms of which the operator solution is constructed and should not be considered as elements of the field algebra intrinsic to the model. Only some combinations of them belong to the intrinsic algebra $\mathfrak{F}_{GNI}$.

The field algebra $\mathfrak{F}_{GNI}$ is a proper subalgebra of $\mathfrak{F}^B_{GNI}$ and is recovered from a particular set of operators constructed from linear combinations and Wick ordered exponentials of linear combinations of Bose fields. Not all Bose fields belong to the algebra $\mathfrak{F}_{GNI}$ of the local fields, nor all vectors of $\mathcal{H}^B_{GNI}$ belong to the state space $\mathcal{H}_{GNI}$ of the GNI formulation of the model. In this way, the set of local states $\mathcal{D}^B_o \equiv \mathfrak{F}^B_{GNI} \Psi_o$, corresponding to the largely redundant field algebra $\mathfrak{F}^B_{GNI}$, contains elements which are not intrinsic to the model. Of course, the Hilbert space $\mathcal{H}_{GNI}$ is a proper subspace of $\mathcal{H}^B_{GNI}$.

Due to the presence of the longitudinal current L^μ in Eq.(2.12), the Gauss' law holds in a weak form and is satisfied on the physical subspace $\mathcal{H}^{phys}_{GNI}$ defined by the subsidiary condition $L^\nu \approx 0$, that is

$$\langle\, \Phi\,,\, (\, J^\nu(x) - \partial_\mu \mathcal{F}^{\mu\nu}(x)\,)\,\Psi\,\rangle \;=\; \langle\, \Phi\,,\, L^\nu(x)\,\Psi\,\rangle \;=\; 0 \quad,\quad \forall\, \Phi\,,\, \Psi \;\in\; \mathcal{H}^{phys}_{GNI}\,. \qquad (2.22)$$

The algebra of the physical operators $\mathfrak{F}^{phys}_{GNI}$ must be identified as the subalgebra of $\mathfrak{F}_{GNI}$ which obeys the subsidiary condition in a proper Hilbert space completion of the local states.

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[9]*A careful and rigorous treatment of the massless scalar field in two dimensions requires the construction of a Krein space in which the set of fields $\{\varphi\} \equiv \{\phi', \chi_2, \lambda'\}$ is realized as a local operator-valued tempered distribution with dense domain $\mathcal{D}_o \equiv \wp(\varphi)\Psi_o$, where $\wp(\varphi)$ is the polynomial algebra generated by the smoothed set of fields $\{\varphi(f)\}$ [17]. In order to associate a Hilbert space of states to the Wightman functions we can follow along the same lines as those of Ref.[11], and find a Krein topology which majorizes the Wightman functions of $\mathfrak{F}_{GNI}$ and defines a maximal Hilbert space structure.*

However, *it is a peculiarity of the anomalous chiral model that the algebra* $\mathfrak{S}_{GNI}^{phys} \equiv \mathfrak{S}_{GNI}$, *since all operators belonging to the intrinsic local field algebra* $\mathfrak{S}_{GNI}$ *commute with the longitudinal current,*

$$[\mathcal{O}, L_\mu] = 0 \ , \ \forall \mathcal{O} \in \mathfrak{S}_{GNI} \ . \tag{2.23}$$

This is expected in the anomalous case since the theory has lost the local gauge invariance. Then the local field algebra $\mathfrak{S}_{GNI}$ *is a singlet under local gauge transformations and is represented in the state space* $\mathcal{H}_{GNI}^{phys} \equiv \mathcal{H}_{GNI}$ *of the* GNI *formulation* [10].

The intrinsic irreducible set of fields $\{\bar{\psi}, \psi, \mathcal{A}_\mu\}$, in terms of which the GNI formulation is defined has been written in terms of the building blocks $\{\chi_1, \chi_2, \phi', \lambda'\}$ by Eqs.(2.19). From this set of intrinsic fields we construct all operators belonging to the algebra $\mathfrak{S}_{GNI}$, as for example the vector current

$$J^\mu = \frac{1}{m_a} \tilde{\partial}^\mu \chi_1 + L^\mu \ , \tag{2.24}$$

and the field-strength tensor

$$\mathcal{F}^{\mu\nu} = m_a \varepsilon^{\mu\nu} \chi_1 \ . \tag{2.25}$$

Since the field χ_1 belongs to $\mathfrak{S}_{GNI}$, i.e.,

$$\chi_1 = \frac{1}{2m_a} \varepsilon^{\mu\nu} \mathcal{F}_{\mu\nu} \ \in \mathfrak{S}_{GNI} \ , \tag{2.26}$$

then, besides the longitudinal current, the Wick exponential : $\exp\{2i\sqrt{\pi/(a-1)}\,\chi_1\}$: $\in$ $\mathfrak{S}_{GNI}$ [11,17]. Therefore the operator

$$W \ \dot{=} \ : \exp\left\{2i\sqrt{\frac{\pi}{a-1}}\,\chi_1\right\}\psi_\ell := \psi_\ell^{\ 0}\Gamma \ , \tag{2.27}$$

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[10] *This does not occur in the VSM, in which neither ψ nor $\mathcal{A}_\mu$ are singlet under local gauge transformations since they do not commute with the constraints. In this genuine gauge theory we have a prehilbert space so that by completions and quotients one gets a physical Hilbert subspace defined by gauge invariant states accommodated as equivalent classes* [13].

also belongs to $\Im_{GNI}$. *In order to be defined on $\mathcal{H}_{GNI}$ the operator W cannot be further reduced.* Γ is a spurious operator with zero scale dimension and spin that generates constant Wightman functions

$$\langle \Gamma^*(x_1)\cdots\Gamma^*(x_m)\Gamma^*(x_{m+1})\cdots\Gamma^*(x_n)\,\Gamma(y_1)\cdots\Gamma(y_m)\Gamma(y_{m+1})\cdots\Gamma(y_n)\rangle = 1 \ , \forall\, a > 1 \ ,$$
(2.28)

and therefore the correlation functions of the operator W are isomorphic to those of the free fermion field ψ_ℓ^0,

$$\langle \prod_{i=1}^n W^*(x_i) \prod_{j=1}^n W(y_j)\rangle \equiv \langle \prod_{i=1}^n \psi_\ell^{*\,0}(x_i) \prod_{j=1}^n \psi_\ell^0(y_j)\rangle \ , \ \forall\, a > 1 \ .$$
(2.29)

The Hilbert space $\mathcal{H}_{GNI}$ of the GNI formulation can be factorized as a product

$$\mathcal{H}_{GNI} = \mathcal{H}_{\chi_1} \otimes \mathcal{H}_{\psi^o, \chi_2, \lambda'} \ ,$$
(2.30)

where $\mathcal{H}_{\chi_1}$ is the Fock space of the free massive field $\chi_1 \in \Im_{GNI}$ and $\mathcal{H}_{\psi^o, \chi_2, \lambda'}$ the closure of the space

$$\mathcal{H}_{\psi^o, \chi_2, \lambda'} = \Im'_{GNI} \ \Psi_o \ ,$$
(2.31)

in which we have a representation of the field algebra $\Im'_{GNI}$ generated by j_ℓ^μ, L^μ, W and the longitudinal piece of the gauge field $\mathcal{A}_L^\mu$ (which only depends on the massless fields); i. e., the closure of the space is obtained by applying to the vacuum polynomials of the fields j_ℓ^μ, W, L^μ and $\mathcal{A}_L^\mu$. Except for the Wick exponential W, the field algebra $\Im'_{GNI}$ is generated by combinations of derivatives of the fields ϕ', χ_2 and λ'. The Hilbert space $\mathcal{H}_{GNI}$ is given by

$$\mathcal{H}_{GNI} \equiv \Im(\chi_1, \Im'_o)_{GNI} \Psi_o \ ,$$
(2.32)

where $\Im(\chi_1, \Im'_o)_{GNI}$ is the field algebra generated by χ_1 and by the subalgebra $\Im'_o \subset \Im'_{GNI}$ which commutes with L^μ.

The operator W cannot be reduced and the Hilbert space completion $\mathcal{H}_{\psi^o, \chi_2, \lambda'}$ cannot be further decomposed. The Hilbert space factorization (2.30) and the isomorphism expressed by Eq.(2.29) imply that the Hilbert space $\mathcal{H}_{GNI}$ contains free fermion states. As we shall see later, the improper factorization of the completion of states performed in Refs.[9,10] leads to some misleading conclusions about basic structural properties of the model.

15

B. Cluster Decomposition

Although the field operator ψ_ℓ representing the fermionic particle in the GNI formulation is given in terms of a spurious operator Γ, the cluster property for the corresponding Wightman functions is not violated. In particular, we obtain for the fermion two-point function [8,12]

$$\langle\, \psi_\ell^*(x)\, \psi_\ell(y)\,\rangle \;=\; \langle\, \psi_\ell^{*\,0}(x)\, \psi_\ell^0(y)\,\rangle\; \exp\left\{\, \frac{4\pi}{a-1}\,\Delta^{(+)}(x-y;m_a)\,\right\}. \tag{2.33}$$

Contrary to what happens in the VSM [13], the space-time contribution coming from the free fermion two-point function in eqs.(2.29-33) ensures the existence of fermions in the asymptotic states and implies that the cluster decomposition is not violated

$$\langle\, \psi_\ell(x+\lambda)\Psi_o\,,\; \psi_\ell(x)\Psi_o\,\rangle_{\lambda\to\infty} \;\longrightarrow\; 0 \;=\; \|\,\langle\,\Psi_o\,,\; \psi_\ell(x)\,\Psi_o\,\rangle\,\|^2\,. \tag{2.34}$$

Of course, we can construct a composite operator belonging to $\Im_{GNI}$ that carries the free fermion chirality,

$$M \;=\; \psi_\ell^*\,\psi_r \;=\; \frac{\mu_o}{4\pi}\;:\,e^{2i\sqrt{\pi}\,\tilde\phi}\,:\;=:\;e^{2i\sqrt{\frac{\pi}{a-1}}\,\chi_1}\,:\,\psi_\ell^{0*}\,\psi_r^0\,\Gamma^* \;\in\; \Im_{GNI}\,. \tag{2.35}$$

However the cluster decomposition is not violated for the corresponding Wightman functions as well :

$$\langle\, M^*(x)\, M(0)\,\rangle \;=\; e^{\frac{4\pi}{a-1}\Delta^{(+)}(x;m_a)}\left(\frac{\mu_o}{2\pi x^2}\right)_{x^2\to-\infty} \;\to\; 0\,. \tag{2.36}$$

From the cluster property (2.34) and (2.36) we conclude that *the screening and confinement aspects exhibited by the CSM for $a > 1$ differ from those of the VSM* [13]. This result is in agreement with the conclusions of Refs.[8,12], and is crucial for the discussion concerning the claimed equivalence of the VSM and CSM defined for $a = 2$.

A peculiar feature of the CSM which differs from the vector case is the fact that *the cluster decomposition property is not violated for Wightman functions that are representations of the field algebra $\Im_{GNI}$ generated from the irreducible set of field operators $\{\bar\psi, \psi, \mathcal{A}_\mu\}$, provided only the intrinsic field algebra is considered without any reduction of the completion of states.* Although the cluster property is violated for the Wightman functions of the operator Γ, this

operator cannot be defined in $\mathcal{H}_{GNI}$ since $[\,\mathcal{Q}_L^5\,,\,\Gamma(x)\,]\,\neq\,0$, where $\mathcal{Q}_L^5$ is the chiral charge associated with the longitudinal current.

C. Connection with the Minimal Bosonization Scheme

Now, let us show that the minimal bosonization scheme used in Ref.[12] is physically equivalent to the enlarged bosonization scheme used in Refs.[8,9], in spite of the fact that the latter introduces an additional redundant field algebra.

The field operator ψ_ℓ given by eq.(2.19c) can be related to the operator solution $\hat{\psi}_\ell$ given in Ref.[12] in the following way: Defining the non-canonical scalar field

$$\sigma(x) \,\doteq\, \frac{1}{\sqrt{a-1}}\,\chi_1(x)\,, \tag{2.37}$$

we can write $\psi_\ell(x)$ as

$$\psi_\ell(x) \,=\, \hat{\psi}_\ell(x)\,\Gamma(x)\,, \tag{2.38}$$

where the fermion field operator in the minimal bosonization scheme is given by [11]

$$\hat{\psi}_\ell(x) \,=:\, e^{-2i\sqrt{\pi}\sigma(x)}\,:\,\psi_\ell^o(x)\,. \tag{2.39}$$

The fact that the GNI operator solutions $\hat{\psi}(x)$ and $\psi(x)$ are related by a superfluous spurious phase operator means that the bosonization scheme used in Refs.[8,9] introduces an extra (unphysical) redundant field algebra, i. e., it introduces more degrees of freedom than those needed for the description of the model. Of course, *these operator solutions are equivalent*

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[11] *The canonical harmonic field $h(x)$ used in Refs.[1,12] is identified with [8]*

$$h_\ell(x) \,\equiv\, \phi_\ell'(x) + \Big(\frac{1}{\sqrt{a-1}}\chi_2(x) - \frac{g}{\sqrt{\pi}}\lambda'(x)\Big)$$

such that, and in accordance with the results obtained in Refs.[1,2,5,6,8,12], there are only two "physical" bosonic degrees of freedom in the anomalous model: the massive field χ_1 and the massless field ϕ', which acts as potential for the free fermionic current.

since they generate the same Wightman functions, provided only the intrinsic field algebra is considered without any reduction of the completion of states.

The Hilbert space $\hat{\mathcal{H}}_{GNI}$ of the minimal scheme can be decomposed as

$$\hat{\mathcal{H}}_{GNI} = \mathcal{H}_\sigma \otimes \mathcal{H}_{\psi^o} \; , \tag{2.40}$$

and the isomorphism expressed in Eq.(2.29) means that *the Wightman functions that are represented on the Hilbert space completion $\mathcal{H}_{\psi^o}$ of the minimal scheme are isomorphic to those which are represented on the Hilbert space completion $\mathcal{H}_{\chi_2, \lambda', \psi_o}$ of the local states of the enlarged GNI formulation.* This ensures the existence of fermions in the asymptotic states [12].

D. The case $a = 2$

This particular case has generated some confusion in the literature, since the corresponding operator algebra exhibits non-trivial and delicate features which might lead to misleading conclusions about structural properties of the model. As we shall see, one can construct a field subalgebra $\Im_{Sch} \subset {}^B\Im_{GNI}^{ext}$ which is isomorphic to the field subalgebra of the VSM but does not belong to the intrinsic field algebra $\Im_{GNI}$ of the anomalous chiral model [13].

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[12] *In the VSM the field $\hat{\lambda}'$ is absent, $\chi_1 \equiv \tilde{\Sigma}$ and $\chi_2 \equiv \tilde{\eta}$. The Hilbert space can be decomposed as [11]*

$$\mathcal{H}_{Sch} = \mathcal{H}_{\tilde{\Sigma}} \otimes \mathcal{H}_{\psi^o, \tilde{\eta}} \; .$$

[13] The introduction of the dual fields $\tilde{\lambda}'$ (or $\tilde{\chi}_2$) enlarges the bose field algebra $\Im_{GNI}^B$ and defines an external algebra ${}^B\Im_{GNI}^{ext} \supset \Im_{GNI}^B$.

For $a = 2$ the Fermi field operator ψ_{ℓ} can be factorized as

$$\psi_{\ell} = \left(\frac{\mu_o}{2\pi}\right)^{1/2} : e^{-2i\sqrt{\pi}\chi_1} : \sigma_{\ell}\,\sigma_r^{\;*} : e^{-2ig\lambda_{\ell}'} : \in \Im_{GNI} \; , \tag{2.41}$$

where σ_r and σ_{ℓ} are spurious operators, and as we shall prove later they do not belong to $\Im_{GNI}$, given by

$$\sigma_r =: e^{-i\sqrt{\pi}\,[\,(\chi_2 + \tilde{\chi}_2\,) - \frac{g}{\sqrt{\pi}}\,(\lambda' + \tilde{\lambda}'\,)\,]} : =: e^{i\sqrt{\pi}L_r} : \; , \tag{2.42a}$$

$$\sigma_{\ell} =: e^{i\sqrt{\pi}\,[\,(\phi' - \tilde{\phi}'\,) + (\chi_2 - \tilde{\chi}_2\,)\,]} :=: e^{i\sqrt{\pi}L_{\ell}} : \; . \tag{2.42b}$$

For $a = 2$, the field λ_{ℓ}' decouples from the longitudinal current L_{ℓ} given by Eq.(2.17d) and there is a broader class of operators belonging to $\Im_{GNI}^{B}$ that satisfy the subsidiary condition. Since in this case the Wick exponential operator $: \exp\{-2ig\lambda_{\ell}'\} :$ commutes with the constraints [14]

$$[\,L_{\mu}\,, : e^{ig(\tilde{\lambda}' - \lambda')} :\,] = 0 \; , \tag{2.43}$$

it is tempting to extract from the operator (2.41) the dependence on the field λ_{ℓ}', as is done in Ref.[9] for the corresponding GI field operator. The resulting operator shares some resemblance with the composite chiral operator of the VSM [9]. With this procedure the authors of Ref.[9] conclude for the need of the Θ-vacuum parametrization in the anomalous model and its equivalence with the VSM. As we shall see, in the formulation in terms of intrinsic field algebra generated from the irreducible set of local field operators, the operators $\sigma_{\ell}\,\sigma_r^{\;*}$, σ_{ℓ}, σ_r and $: e^{ig(\tilde{\lambda}' \mp \lambda')} :$ do not exist separately and cannot be defined on $\mathcal{H}_{GNI}$.

In order to show that some misleading conclusion can arise by making the factorization of the closure space, as for example $\mathcal{H}_{\psi^o, \chi_2, \lambda'} = \mathcal{H}_{\psi^o, \chi_2, \lambda_r'} \otimes \mathcal{H}_{\lambda_{\ell}'}$ done in Ref.[9], we follow along the same lines as those of Ref.[9] and consider the operator $\psi_{\ell}'(x) \in \Im_{GNI}^{B}$ defined for $a > 1$ and given by

$$\psi_{\ell}'(x) \doteq: e^{+ig[\lambda'(x) - \tilde{\lambda}'(x)]} \psi_{\ell}(x) : \notin \Im_{GNI} \; , \tag{2.44}$$

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[14] *Where we used* $\lambda_{\ell} = 1/2\,(\lambda' - \tilde{\lambda}')$.

Then, we construct the composite operator carrying free chirality

$$S = \psi_\ell'^* \, \psi_r \ .$$ (2.45)

The operator $S \in \mathfrak{S}_{GNI}^{B}$ is the GNI counterpart of the operator which was considered in Ref.[9] in the case $a = 2$ and the corresponding two-point function is given by

$$\langle \, S^*(x) \, S(0) \, \rangle \ = \ e^{\frac{4\pi}{a-1}\Delta^{(+)}(x; m_a)} \, \frac{\mu_0}{2\pi} \Big(x^2 \Big)^{-(a-2)/(a-1)} \ .$$ (2.46)

For $a = 2$ the cluster decomposition is violated ! Of course, this implies a misleading conclusion about the need of the Θ-vacuum parametrization in the GNI formulation of the theory. *The state $\psi_\ell' \, \Psi_o$ belongs to the improper Hilbert space completion $\mathcal{H}_{\psi^o, \, \chi_2, \, \lambda_r'} \subset \mathcal{H}_{GNI}^{B}$ of the local states and cannot be regarded as a state in the Hilbert space of states which defines the representation of the field algebra $\mathfrak{S}_{GNI}$. As we shall see, the set of local states $\mathcal{D}_o \equiv \mathfrak{S}_{GNI} \, \Psi_o$ does not contain states like $\psi_\ell' \, \Psi_o$.*

Note that the introduction of the dual field $\tilde{\lambda}'$ (or $\tilde{\chi}_2$) enlarges the bose field algebra $\mathfrak{S}_{GNI}^{B}$ and defines an external algebra ${}^{B}\mathfrak{S}_{GNI}^{ext} \supset \mathfrak{S}_{GNI}^{B}$. The need of the Θ-vacuum parametrization, claimed in Ref.[9], cannot be regarded as a structural property of the theory since it emerges as a consequence of the use of the redundant algebra belonging to the external algebra ${}^{B}\mathfrak{S}_{GNI}^{ext}$, generated by the set of Bose fields $\{\phi', \tilde{\phi}', \chi_2, \tilde{\chi}_2, \lambda', \tilde{\lambda}', \chi_1\}$, rather than on the intrinsic algebra $\mathfrak{S}_{GNI}$, generated from the irreducible set of field operators which defines the model.

We must remark that on the enlarged state space $\mathcal{H}_{GNI}^{B}$, the connection with the VSM can be made by considering the operator $S_{Sch} \doteq \psi_\ell' \, |_{a=2}$

$$S_{Sch} \doteq \, : e^{+2 \, i \, g \, \lambda_\ell'} \, \psi_r \, : \, |_{a=2} \ = \ \Big(\frac{\mu_o}{2\pi} \Big)^{1/2} \, : e^{-2i\sqrt{\pi}\chi_1} : \, \sigma_\ell \, \sigma_r^* \ \notin \ \mathfrak{S}_{GNI} \ ,$$ (2.47)

such that the cluster property is violated for the corresponding Wightman function

$$\langle \, S_{Sch}^*(x) \, S_{Sch}(0) \, \rangle_{\, x^2 \to -\infty} \ \longrightarrow \ \frac{\mu_o}{2\pi} \ .$$ (2.48)

The operator $S_{Sch} \notin \mathfrak{S}_{GNI}$ and can be viewed as a composite operator

$$S_{Sch}(x) \ = \ \lim_{\epsilon \to 0} \mathcal{Z}^{-1}(\epsilon) \, \Psi_r^*(x + \epsilon) \, \Psi_\ell(x) \ ,$$ (2.49)

20

where $\Psi \notin \Im_{GNI}$ and is given by

$$\Psi(x) = \left(\frac{\mu_o}{2\pi}\right)^{1/4} : e^{i\sqrt{\pi}\gamma^5\chi_1(x)} : e^{i\sqrt{\pi}[\gamma^5\tilde{L}(x)+L(x)]} : . \qquad (2.50)$$

Although the operator $\Psi(x)$ generates the same Wightman functions of the covariant solution of the VSM, it is not the intrinsic field operator representing the fermionic content of the CSM and cannot be defined on $\mathcal{H}_{GNI}$. The operators (2.47) and (2.50) are the gauge noninvariant counterpart of the field operators used in Refs.[9,10] to suggest the need of Θ-vacuum parametrization and the equivalence of the VSM and CSM defined for $a = 2$. Since these operators do not belong to the field algebra $\Im_{GNI}$, they cannot be defined on $\mathcal{H}_{GNI}$ and the claimed equivalence with the VSM cannot be established. This apparent equivalence is a by-product of the incorrect choice of the observables representing the fermionic content of the model, that arises by factorizing the closure of the space as the product $\mathcal{H}_{\psi^o, \chi_2, \lambda'} = \mathcal{H}_{\psi^o, \chi_2, \lambda'_r} \otimes \mathcal{H}_{\lambda'_\ell}$. Note that, in the VSM [11], $\chi_2 \equiv \tilde{\eta}, \chi_1 \equiv \tilde{\Sigma}$ and the closure of the space is $\mathcal{H}_{\psi^o, \tilde{\eta}}$ and cannot be factorized as $\mathcal{H}_{\psi^o, \tilde{\eta}} = \mathcal{H}_{\tilde{\eta}} \otimes \mathcal{H}_{\psi_o}$. The Hilbert space of the VSM can be identified with the improper subspace $\mathcal{H}_{Sch} \equiv \mathcal{H}_{\chi_1} \otimes \mathcal{H}_{\psi^o_\ell, \chi_2, \lambda'_r}$ of the chiral model defined for $a = 2$.

Now we prove the proposition according to which one cannot define on the Hilbert space $\mathcal{H}_{GNI}$ the Wick exponential

$$\Lambda_\ell(x) \doteq : e^{ig[\lambda'(x) - \tilde{\lambda}'(x)]} : , \qquad (2.51)$$

and therefore the operator $\psi'_\ell \doteq : \Lambda_\ell \psi_\ell :$ also cannot be defined on $\mathcal{H}_{GNI}$. In analogy with the vector case [11], such property follows from the fact that some charges get trivialized in the restriction from $^B\mathcal{H}^{ext}_{GNI}$ or from $\mathcal{H}^B_{GNI}$ to $\mathcal{H}_{GNI}$.

For $a \neq 2$, the trivialization of the chiral charge $\mathcal{Q}^5_L$ in the restriction from $\mathcal{H}^B_{GNI}$ to $\mathcal{H}_{GNI}$,

$$\mathcal{Q}^5_L \mathcal{H}^B_{GNI} \neq 0 , \quad \mathcal{Q}^5_L \mathcal{H}_{GNI} = 0 , \qquad (2.52)$$

implies that the closure of local states associated to the intrinsic field algebra $\Im_{GNI}$ does not allow the introduction of operators which are charged under $\mathcal{Q}^5_L$. However, for the especial

case $a = 2$, the field operators $\Lambda_\ell(x)$ and $\psi'_\ell(x)$ are neutral under $\mathcal{Q}_L^5$,

$$[\, \mathcal{Q}_L^5 \,, \Lambda_\ell(x) \,] \, = \, 0 \,, \tag{2.53}$$

$$[\, \mathcal{Q}_L^5 \,, \psi'_\ell(x) \,] \, = \, 0 \,, \tag{2.54}$$

and the criterion based on the trivialization of the charge $\mathcal{Q}_L^5$ is insufficient to decide about the existence of the state $\psi'_\ell \Psi_o$ on $\mathcal{H}_{GNI}$.

Consider a local charge operator

$$\mathcal{Q}_{\tilde{\varphi}} \, \doteq \, \lim_{R \to \infty} \int_{|x^1| \leq R} \partial_{x^0} \tilde{\varphi}(x^0, x^1) \, dx^1 \, \equiv \, \lim_{R \to \infty} \mathcal{Q}_{\tilde{\varphi}, R} \tag{2.55}$$

associated with the field operator $\tilde{\varphi}(x)$ [15] such that it get trivialized in the restriction from $^B\mathcal{H}_{GNI}^{ext}$ or from $\mathcal{H}_{GNI}^B$ to $\mathcal{H}_{GNI}$, i.e.,

$$\mathcal{Q}_{\tilde{\varphi}} \, ^B\mathcal{H}_{GNI}^{ext} \, \neq \, 0 \quad , \quad \begin{cases} \mathcal{Q}_{\tilde{\varphi}} \, \mathcal{H}_{GNI}^B \, \neq \, 0 \\ \\ or \\ \\ \mathcal{Q}_{\tilde{\varphi}} \, \mathcal{H}_{GNI}^B \, = \, 0 \end{cases} \quad , \quad \mathcal{Q}_{\tilde{\varphi}} \, \mathcal{H}_{GNI} \, = \, 0 \,. \tag{2.56}$$

The trivialization of the charge $\mathcal{Q}_{\tilde{\varphi}}$ on the Hilbert space $\mathcal{H} \, \doteq \, \Im \, \Psi_o$, follows from the fact that [11], if the field algebra $\Im$ contains functions of φ but not of $\tilde{\varphi}$, then

$$\langle \, \mathcal{Q}_{\tilde{\varphi}} \Psi_o \,, \Im \, \Psi_o \, \rangle \, = \, 0 \,, \tag{2.57}$$

and hence, we get

$$\lim_{R \to \infty} \langle \, \mathcal{Q}_{\tilde{\varphi}, R} \, \Im \, \Psi_o \,, \Im \, \Psi_o \, \rangle \, =$$

$$= \, \lim_{R \to \infty} \langle \, [\, \mathcal{Q}_{\tilde{\varphi}, R} \,, \Im \,] \, \Psi_o \,, \Im \, \Psi_o \, \rangle \, + \, \lim_{R \to \infty} \langle \, \Im \, \mathcal{Q}_{\tilde{\varphi}, R} \, \Psi_o \,, \Im \, \Psi_o \, \rangle \, = \, 0 \,, \tag{2.58}$$

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[15] *In which a regularization of the integral is to be understood [11],*

$$\mathcal{Q}_{\tilde{\varphi}, R} \, = \, \int \partial_{x^0} \tilde{\varphi}(x^0, x^1) \, f_R(x^1) \, g(x^0) \, dx^1 \, dx^0 \,,$$

with $f_R = f(|x^1|/R)$, $f \in \mathcal{D}(\Re^1)$, $f(x) = 1$ for $|x| < 1$, $f(x) = 0$ for $|x| > 1 + \epsilon$, $g(x^0) \in \mathcal{D}(\Re^1)$, $\int g(x^0) dx^0 = 1$.

i.e., for the set of local states $\mathcal{D}_o \equiv \mathfrak{I}_{GNI}\,\Psi_o$, we obtain the weak limit

$$\mathcal{Q}_{\tilde{\varphi}}\,\mathcal{D}_o \;=\; w - \lim_{R\to\infty}\,\mathcal{Q}_{\tilde{\varphi},R}\,\mathcal{D}_o \;=\; 0\;. \tag{2.59}$$

Now, we prove that the trivialization of the charge $\mathcal{Q}_{\tilde{\lambda}'}$ in the restriction from $^B\mathcal{H}^{ext}_{GNI}$ to $\mathcal{H}_{GNI}$ implies that the state $\psi'_\ell(x)\,\Psi_o$ cannot belong to $\mathcal{H}_{GNI}$.

To this end, let $A(\tilde{\varphi})$ be an element of the local field algebra $\mathfrak{I}^B_{GNI}$ (is a local operator) and that have a non zero charge $\mathcal{Q}_{\tilde{\varphi}}$,

$$\lim_{R\to\infty}\,[\,\mathcal{Q}_{\tilde{\varphi},R}\,,\,A(\tilde{\varphi})\,] \;=\; \alpha\,A(\tilde{\varphi})\;,\quad \alpha\neq 0\;,\quad A(\tilde{\varphi})\in\mathfrak{I}^B_{GNI}\;. \tag{2.60}$$

For the set of local states $\mathcal{D}_o \equiv \mathfrak{I}_{GNI}\,\Psi_o$, and a local state $A(\tilde{\varphi})\Psi_o$ of charge α, consider

$$\alpha\,\langle\,A(\tilde{\varphi})\Psi_o\,,\,\mathfrak{I}_{GNI}\,\Psi_o\,\rangle \;=\; \langle\,[\,\mathcal{Q}_{\tilde{\varphi}}\,,\,A(\tilde{\varphi})\,]\,\Psi_o\,,\,\mathfrak{I}_{GNI}\,\Psi_o\,\rangle \;=\;$$

$$=\; \langle\,A(\tilde{\varphi})\,\Psi_o\,,\,\mathcal{Q}_{\tilde{\varphi}}\,\mathfrak{I}_{GNI}\,\Psi_o\,\rangle \;-\; \langle\,A(\tilde{\varphi})\,\mathcal{Q}_{\tilde{\varphi}}\,\Psi_o\,,\,\mathfrak{I}_{GNI}\,\Psi_o\,\rangle \;=\; 0\;, \tag{2.61}$$

and therefore the state $A(\tilde{\varphi})\,\Psi_o$ does not exist in $\mathcal{H}_{GNI}$.

Considering $\tilde{\varphi}\equiv\tilde{\lambda}'$, Eq.(2.61) implies that *the closure of the local states associated to the field algebra* $\mathfrak{I}_{GNI}$ *intrinsic to the model does not allow the introduction of operators which are charged under* $\mathcal{Q}_{\tilde{\lambda}'}$. Since

$$[\,\mathcal{Q}_{\tilde{\lambda}'}\,,\,\Lambda_\ell(x)\,] \;=\; \frac{\pi}{g}\,\Lambda_\ell(x)\;, \tag{2.62a}$$

$$[\,\mathcal{Q}_{\tilde{\lambda}'}\,,\,\psi'_\ell(x)\,] \;=\; \frac{\pi}{g}\,\psi'_\ell(x)\;, \tag{2.62b}$$

the set of local states $\mathcal{D}_o \equiv \mathfrak{I}_{GNI}\,\Psi_o$ *does not contain states like* $\Lambda_\ell(x)\,\Psi_o$ *and* $\psi'_\ell(x)\,\Psi_o$. The state $\psi'_\ell\,\Psi_o$ cannot belong to $\mathcal{H}_{GNI}$ and the operator $\psi'_\ell\notin\mathfrak{I}_{GNI}$.

Similarly, for $\tilde{\varphi}\equiv\tilde{\chi}_2\in{}^B\mathfrak{I}^{ext}_{GNI}$ and the trivialization of the charge $\mathcal{Q}_{\tilde{\chi}_2}$ in the restriction from $^B\mathcal{H}^{ext}_{GNI}$ to $\mathcal{H}_{GNI}$, one proves that the operators σ_ℓ and σ_r which are charged under $\mathcal{Q}_{\tilde{\chi}_2}$, do not exist separately in $\mathfrak{I}_{GNI}$ and cannot be defined on $\mathcal{H}_{GNI}$. Only the neutral combination W exist in the field algebra $\mathfrak{I}_{GNI}$.

In conclusion we showed that the main results of Refs.[9,10] cannot be regarded as structural properties of the CSM since they are consequence of the use of a redundant field algebra belonging to the external algebra $\mathfrak{I}^B$ rather than on the use of the intrinsic field algebra $\mathfrak{I}$ which defines the model.

III. GAUGE INVARIANT FORMULATION

In the Lagrangian formalism and within the Faddeev-Satashvili proposal [3], the so-called GI formulation of the anomalous gauge theory is constructed introducing extra degrees of freedom into the theory by adding to the original GNI Lagrangian a WZ term [3,4]

$$\mathcal{L}_{GI} = \mathcal{L}_{GNI} + \mathcal{L}_{WZ} , \tag{3.1}$$

where $\mathcal{L}_{WZ}$ is the WZ Lagrangian density given by

$$\mathcal{L}_{WZ} = \frac{1}{2}(a-1)(\partial_\mu \theta)^2 + \frac{g}{\sqrt{\pi}}\mathcal{A}_\mu\{(a-1)\partial^\mu\theta - \tilde{\partial}^\mu\theta\} . \tag{3.2}$$

The resulting "embedded' theory exhibits invariance under the *extended* local gauge transformations

$$\psi(x) \rightarrow e^{i\Lambda(x)P_+}\psi(x) , \tag{3.3a}$$

$$\mathcal{A}_\mu(x) \rightarrow \mathcal{A}_\mu(x) + \frac{1}{g}\partial_\mu\Lambda(x) , \tag{3.3b}$$

$$\theta(x) \rightarrow \theta(x) + \Lambda(x) . \tag{3.3c}$$

Within the path-integral and operator approaches, the GI formulation is constructed by enlarging the intrinsic local field algebra $\mathfrak{I}_{GNI}$ through an operator-valued gauge transformation on the GNI formulation of the anomalous model [5,6,7],

$$^\theta\psi(x) =: e^{-i\sqrt{\pi}\theta(x)P_+}\psi(x) : , \tag{3.4a}$$

$$^\theta\mathcal{A}_\mu(x) = \mathcal{A}_\mu(x) - \frac{\sqrt{\pi}}{g}\partial_\mu\theta(x) . \tag{3.4b}$$

The gauge-transformed Lagrangian density is given by [7]

$$\mathcal{L}_{GI} = -\frac{1}{4}(\mathcal{F}_{\mu\nu})^2 + i\,^\theta\bar{\psi}\,^\theta\!\!\!\not{\partial}\,^\theta\psi + \frac{g}{\sqrt{\pi}}\,^\theta\mathcal{K}_\ell^\mu\,^\theta\mathcal{A}_\mu + \frac{ag^2}{2\pi}(^\theta\mathcal{A}_\mu)^2 , \tag{3.5}$$

which can be written as

$$\mathcal{L}_{GI}\{^\theta\bar{\psi},\,^\theta\psi,\,^\theta\mathcal{A}_\mu\} = \mathcal{L}_{GNI}\{\bar{\psi},\psi,\mathcal{A}_\mu\} + \mathcal{L}_{WZ}\{\mathcal{A}_\mu,\theta\} . \tag{3.6}$$

The bosonized effective GI Lagrangian density is given by (2.5) plus the WZ Lagrangian (3.2).

The set of gauge-transformed field operators $\{{}^{\theta}\bar{\psi}, {}^{\theta}\psi, {}^{\theta}\mathcal{A}_{\mu}\}$ is invariant (by construction) under the extended local gauge transformations (3.3) and generates the GI field algebra ${}^{\theta}\Im \equiv \Im_{GI}$.

Within the formulation of the theory based on the local field algebra generated from the intrinsic irreducible set of field operators $\{\bar{\psi}, \psi, \mathcal{A}^{\mu}\}$, the procedure of defining the GI version of the anomalous theory make use of the introduction of a Bose field algebra, generated by the WZ field θ, which is an external algebra to the intrinsic field algebra $\Im_{GNI}$. We shall denote by ${}^{\theta}\mathcal{H} \equiv \mathcal{H}_{GI} \doteq {}^{\theta}\Im \Psi_{o}$, the state space on which the field algebra generated from the set of gauge-transformed field operators $\{{}^{\theta}\bar{\psi}, {}^{\theta}\psi, {}^{\theta}\mathcal{A}_{\mu}\}$ is represented.

The question that arises refers to whether the introduction of the external WZ field algebra changes the physical content of the original anomalous field theory. Without neglecting the required mathematical rigor, the algebraic isomorphism between the GNI and GI formulations of the anomalous chiral model can be established starting from the enlargement of the intrinsic field algebra $\Im_{GNI}$. As we will show the introduction of the WZ field enlarges the theory and replicates it, without changing either its algebraic structure or its physical content. As a matter of fact, the implementation of the operator-valued gauge transformation connecting the GNI and GI formulations is undercover of the algebraic isomorphism between the corresponding field algebras. This is most easily understood in the bosonization approach in which the field algebras $\Im_{GNI}^{B}$ and $\Im_{GNI}$ are available.

To begin with and following along the same lines as those of Refs.[8,9], we consider the enlargement of the Bose field algebra $\Im_{GNI}^{B}$ by the introduction of the WZ field through

$$\theta = \xi' - \frac{g}{\sqrt{\pi}}\left(\frac{1}{a-1}\chi + \lambda\right), \tag{3.7}$$

which corresponds to shift the field λ' by

$$-\frac{g}{\sqrt{\pi}}\lambda' = \theta - \xi', \tag{3.8}$$

25

with the fields satisfying the algebraic constraints

$$[\,\lambda'(x)\,,\,\xi'(y)\,] = 0\,, \tag{3.9a}$$

$$[\,\theta(x)\,,\,\xi'(y)\,] = -\frac{g}{\sqrt{\pi}}\,[\,\theta(x)\,,\,\lambda'(y)\,] = [\,\xi'(x)\,,\,\xi'(y)\,] = \frac{g^2}{\pi}\,[\,\lambda'(x)\,,\,\lambda'(y)\,]\,. \tag{3.9b}$$

As we shall see, the shift (3.8) together with the commutations relations above leads to a gauge invariant algebra and ensures that no additional physical degree of freedom is introduced into the theory.

With the field re-definition (3.8) we introduce a Bose field subalgebra $\Im^{B}_{GI} \subset \Im^{B}_{GNI}$, generated by the set of building blocks $\{\,\phi',\chi_2,\chi_1,\xi'\,\}$, and the Hilbert space $\mathcal{H}^{B}_{GNI} \doteq \Im^{B}_{GNI}\,\Psi_o$ can be decomposed as

$$\mathcal{H}^{B}_{GNI} = \mathcal{H}_{\chi_1} \otimes \mathcal{H}_{\phi'} \otimes \mathcal{H}_{\chi_2} \otimes \mathcal{H}_{\lambda'} = \mathcal{H}_{\chi_1} \otimes \mathcal{H}_{\phi'} \otimes \mathcal{H}_{\chi_2} \otimes \mathcal{H}_{\xi'} \otimes \mathcal{H}_{\theta} \equiv \mathcal{H}^{B}_{GI} \otimes \mathcal{H}_{\theta}\,, \tag{3.10}$$

where the subspace $\mathcal{H}^{B}_{GI} \subset \mathcal{H}^{B}_{GNI}$ is defined by $\mathcal{H}^{B}_{GI} \doteq \Im^{B}_{GI}\,\Psi_o$.

Using (3.8), the set of field operators $\{\bar{\psi}_{GNI},\psi_{GNI},\mathcal{A}^{\mu}_{GNI}\}$ given by Eq.(2.19), and which defines the GNI algebra $\Im_{GNI}$, can be written as

$$\psi_{GNI}(x) =:\psi_{GI}(x)\,e^{\,i\sqrt{\pi}\,P_+\,\theta(x)} := \mathcal{Z}^{1/2}\,\psi_{GI}(x) : e^{\,i\sqrt{\pi}\,P_+\,\theta(x)} :\,, \tag{3.11}$$

$$\mathcal{A}^{\mu}_{GNI}(x) = \mathcal{A}^{\mu}_{GI}(x) + \frac{\sqrt{\pi}}{g}\,\partial^{\mu}\,\theta(x)\,, \tag{3.12}$$

where

$$\mathcal{Z} = \exp\{\,4\pi\,\langle\,\theta(x+\epsilon)\,\xi'(x)\,\rangle_o\,\} = [\,-\mu\,\epsilon^2\,]^{1/(a-1)} \tag{3.13}$$

is a finite wave function renormalization constant $(a > 1)$, and the longitudinal current can be written as

$$L^{\mu}_{GNI}(x) = L^{\mu}_{GI}(x) - \mathcal{J}^{\mu}_{WZ}(x)\,, \tag{3.14}$$

where the WZ current is

$$\mathcal{J}^{\mu}_{WZ} = -\frac{g}{\sqrt{\pi}}\,\{\,(\,a-1\,)\,\partial^{\mu}\theta - \tilde{\partial}^{\mu}\theta\,\}\,, \tag{3.15}$$

and

$$\langle\,L^{\mu}_{GI}\,\Psi_o\,,\,L^{\nu}_{GI}\,\Psi_o\,\rangle = 0\,. \tag{3.16}$$

26

The anomalous nature of the model induces the transformation (3.14) for the longitudinal piece of the current [7], such that the Hilbert space $\mathcal{H}_{GI} \doteq \mathfrak{I}_{GI} \Psi_o$ is defined by the subsidiary condition

$$\langle \Phi, (\, \mathcal{J}^{\nu}_{GI}(x) - \partial_{\mu} \mathcal{F}^{\mu\nu}(x)) \Psi \rangle = \langle \Phi, L^{\nu}_{GI}(x) \Psi \rangle = 0 \quad, \quad \forall \Phi, \Psi \in \mathcal{H}_{GI}, \quad (3.17)$$

with

$$^{\theta} \mathcal{J}^{\mu} \equiv \mathcal{J}^{\mu}_{GI}(x) = \mathcal{J}^{\mu}_{GNI}(x) + \mathcal{J}^{\mu}_{WZ}(x). \quad (3.18)$$

Using (3.8) together with (2.17b), yields

$$\phi_{\ell} - \theta = \frac{1}{2}(\phi - \tilde{\phi}) = \phi'_{\ell} + \frac{1}{\sqrt{a-1}}(\chi_2 - \chi_1) - \xi'. \quad (3.19)$$

Note that the expression for the field combination $\phi_{\ell} - \theta$ is the same as that for ϕ_{ℓ} given by eq.(2.17b), but with the non canonical free field λ' replaced by ξ'. In agreement with one of the main conclusions of Ref.[8], we find that *the field algebra* $\mathfrak{I}^{B}_{GNI}$ *of the GNI formulation generated from the building blocks* $\{\phi', \chi_2, \chi_1, \lambda'\}$ *is replaced in the GI formulation by the algebra* $\mathfrak{I}^{B}_{GI}$ *generated from the set of fields* $\{\phi', \chi_2, \chi_1, \xi'\}$.

Using the transformations (2.10) and (3.8), we obtain the bosonized version of the *GI* Lagrangian as given by [8]

$$\mathcal{L}_{GI} = \frac{1}{2}(\partial_{\mu}\phi')^2 - \frac{1}{2}(\partial_{\mu}\chi_2)^2 + \frac{(a-1)}{2}(\partial_{\mu}\xi')^2 + \frac{1}{2}(\partial_{\mu}\chi_1)^2 - \frac{1}{2}m_a^2 \chi_1^2. \quad (3.20)$$

The formal expressions for the operators $\{\psi_{GI}, \mathcal{A}^{\mu}_{GI}, L^{\mu}_{GI}\}$ are the same as for $\{\psi_{GNI}, \mathcal{A}^{\mu}_{GNI}, L^{\mu}_{GNI}\}$, except for the replacement of the field $g/\sqrt{\pi}\lambda'$ by the field ξ'. The set of field operators $\{\bar{\psi}_{GI}, \psi_{GI}, \mathcal{A}^{\mu}_{GI}\}$ defines (through polynomials of these smeared fields, Wick ordering, point-splitting regularization of polynomials, etc.) the *GI* algebra $\mathfrak{I}_{GI}$ which is subject to the constraint

$$[\mathcal{O}, L^{\mu}_{GI}] = 0 \quad, \forall \mathcal{O} \in \mathfrak{I}_{GI}. \quad (3.21)$$

Although the so-introduced WZ field has acquired dynamics, it is a redundant field in the sense that it does not change the algebraic structure of the model and therefore does

not change its physical content. As we shall see, this implies the isomorphism between the field algebras $\Im_{GNI}$ and $\Im_{GI}$.

From the algebraic point of view, the WZ field introduced according with (3.8) and subjected to the commutation relations (3.9), plays a redundant role on the structure of the Bose field algebra $\mathcal{H}^{B}_{GNI}$. This can be seen by considering the Wightman functions associated with the Wick exponential of the field λ':

$$\langle \prod_{j=1}^{n} : e^{-i\,g\,\lambda'(x_j)} : \prod_{k=1}^{n} : e^{i\,g\,\lambda'(y_k)} : \rangle = \langle \prod_{j=1}^{n} : e^{i\sqrt{\pi}\,[\theta(x_j)-\xi'(x_j)]} : \prod_{k=1}^{n} : e^{-i\sqrt{\pi}\,[\theta(y_k)-\xi'(y_k)]} : \rangle =$$

$$= 1 \times \langle \prod_{j=1}^{n} : e^{-i\sqrt{\pi}\,\xi'(x_j)} : \prod_{k=1}^{n} : e^{i\sqrt{\pi}\,\xi'(y_k)} : \rangle \equiv$$

$$\equiv \langle \prod_{j=1}^{n} : e^{-i\,[\,g\,\lambda'(x_j)-\sqrt{\pi}\,\theta(x_j)]} : \prod_{k=1}^{n} : e^{i\,[\,g\,\lambda'(y_k)-\sqrt{\pi}\,\theta(y_k)]} : \rangle \,, \qquad (3.22)$$

in which we have explicited the identity arising from the commutator factors

$$1 = \exp\Big\{ \pi \sum_{j,\,k\,=\,1}^{n} \Big(\langle\,\theta(x_j)\,\theta(y_k)\,\rangle_o - \langle\,\theta(x_j)\,\xi'(y_k)\,\rangle_o - \langle\,\xi'(x_j)\,\theta(y_k)\,\rangle_o \Big) +$$

$$+ \pi \sum_{i<j}^{n} \Big(\langle\,\xi'(x_i)\,\theta(x_j)\,\rangle_o + \langle\,\theta(x_i)\,\xi'(x_j)\,\rangle_o - \langle\,\theta(x_i)\,\theta(x_j)\,\rangle_o \Big) +$$

$$+ \pi \sum_{k<l}^{n} \Big(\langle\,\xi'(y_k)\,\theta(y_l)\,\rangle_o + \langle\,\theta(y_k)\,\xi'(y_l)\,\rangle_o - \langle\,\theta(y_k)\,\theta(y_l)\,\rangle_o \Big) \Big\} \,. \qquad (3.23)$$

As a consequence of the identity (3.22) we obtain

$$\langle \prod_{j=1}^{n} \psi^{\dagger}_{GNI}(x_i) \prod_{k=1}^{n} \psi_{GNI}(y_k) \rangle \equiv \langle \prod_{j=1}^{n} \psi^{\dagger}_{GI}(x_i) \prod_{k=1}^{n} \psi_{GI}(y_k) \rangle \,, \qquad (3.24)$$

expressing the isomorphism between the Wightman functions generated from the field operator $\psi_{GNI} \in \Im_{GNI}$ and those generated from the field operator $\psi_{GI} \in \Im_{GI}$. In the same way, we get

$$\langle \prod_{j=1}^{n} \mathcal{A}^{\mu}_{GNI}(x_i) \prod_{k=1}^{n} \mathcal{A}^{\nu}_{GNI}(y_k) \rangle \equiv \langle \prod_{j=1}^{n} \mathcal{A}^{\mu}_{GI}(x_i) \prod_{k=1}^{n} \mathcal{A}^{\nu}_{GI}(y_k) \rangle \,. \qquad (3.25)$$

From the above considerations, the isomorphism between the field algebra $\Im_{GNI}$ defining the GNI formulation and the algebra $\Im_{GI}$ defining the GI formulation of the CSM implies that the state space $\mathcal{H}_{GNI}$, which provides a representation of the GNI intrinsic local field

algebra $\Im_{GNI}$, is isomorphic to the state space $\mathcal{H}_{GI}$ on which the GI local field algebra is represented; i.e.,

$$\langle \wp \{ \bar{\psi}_{GNI},\, \psi_{GNI},\, \mathcal{A}^{\nu}_{GNI} \} \rangle \equiv \langle \wp \{ \bar{\psi}_{GI},\, \psi_{GI},\, \mathcal{A}^{\nu}_{GI} \} \rangle \equiv \langle \wp \{ ^{\theta}\bar{\psi},\, ^{\theta}\psi,\, ^{\theta}\mathcal{A}^{\nu} \} \rangle \,, \qquad (3.26)$$

where $\wp$ is any polynomial in the intrinsic field operators.

The fact that the Wightman functions of the GNI formulation are the same of those of GI formulation is in agreement with the results of Refs.[6,8] in the sense that *the introduction of the WZ field enlarges the theory and replicates it, without changing neither its algebraic structure nor its physical content.* Consequently the same analysis made in section 2 for the GNI formulation applies to its isomorphic GI formulation. The conclusions referring to cluster decomposition and Θ-vacuum parametrization and the suggested equivalence of VSM and CSM defined for $a = 2$, are the same for the GI formulation. Therefore we completely disagree with the conclusion of the authors of Refs.[9,10] who claim the need for a Θ-vacuum parametrization and the equivalence of the VSM and the GI formulation of CSM defined for $a = 2$.

Now, let us make some final remarks on the suggested connection with the VSM. In Ref.[9] it is claimed that for $a = 2$ and apart from free physical chiral states, the GI formulation of CSM is equivalent to the VSM. These statements disagree with our conclusions [16] about the behavior of the fermionic particle in both GNI and GI formulations and are misleading in view of the algebraic isomorphism between GNI and GI formulations.

The local charge operator $\mathcal{Q}_{\varphi} = \lim_{R \to \infty} \mathcal{Q}_{\varphi,R}$, associated with a field operator φ, defines automorphisms of the field algebras $^{B}\Im^{ext}_{\alpha}$, $\Im^{B}_{\alpha}$ and $\Im_{\alpha}$, which are implementable in the various spaces $^{B}\mathcal{H}^{ext}_{\alpha}$, $\mathcal{H}^{B}_{\alpha}$ and $\mathcal{H}_{\alpha}$. Using the algebraic constraints (3.9), we can display the following implementability conditions

$$\mathcal{Q}_{\lambda'} \mathcal{H}^{B}_{GNI} \neq 0 \,, \ \ \mathcal{Q}_{\lambda'} \mathcal{H}_{GNI} \neq 0 \,,$$

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[16] *And also disagrees with the conclusions of Refs.[6,8] and [12].*

$$\mathcal{Q}_{\lambda'}\mathcal{H}_{GI}^{B} = 0 \ , \ \mathcal{Q}_{\lambda'}\mathcal{H}_{GI} = 0 \ ; \tag{3.27a}$$

$$\mathcal{Q}_{\xi'}\mathcal{H}_{GNI}^{B} = 0 \ , \ \mathcal{Q}_{\xi'}\mathcal{H}_{GNI} = 0 \ ,$$

$$\mathcal{Q}_{\xi'}\mathcal{H}_{GI}^{B} \neq 0 \ , \ \mathcal{Q}_{\xi'}\mathcal{H}_{GI} \neq 0 \ ; \tag{3.27b}$$

and since $\theta = \xi' - \frac{g}{\sqrt{\pi}}\lambda'$, we get

$$\mathcal{Q}_{\theta}\mathcal{H}_{GNI}^{B} \neq 0 \ , \ \mathcal{Q}_{\theta}\mathcal{H}_{GNI} \neq 0 \ ,$$

$$\mathcal{Q}_{\theta}\mathcal{H}_{GI}^{B} \neq 0 \ , \ \mathcal{Q}_{\theta}\mathcal{H}_{GI} \neq 0 \ . \tag{3.27c}$$

The latter condition implies that the Wick exponential and derivatives of the WZ field θ can be defined on both Hilbert spaces $\mathcal{H}_{GNI}$ and $\mathcal{H}_{GI}$, as implied by the operator-valued gauge transformation (3.4) which enables the relationship between the isomorphic GNI and GI formulations.

Following along the same lines as those of section 2.4 for the GNI formulation, for $a = 2$ the Fermi field operator $\psi_{\ell GNI}$ can be written as

$$\psi_{\ell GNI} =: \psi'_{\ell GI} \ e^{2i\sqrt{\pi}\theta_r} :: e^{2i\sqrt{\pi}(\theta_\ell - \xi'_\ell)} : \ , \tag{3.28}$$

where the expression for the operator $\psi'_{\ell GI}$ is the same of Eq.(2.47), only replacing the field λ'_r by the field ξ'_r.

The trivialization of the charge $\mathcal{Q}_{\tilde{\lambda}'}$, carried by the field combination $\theta_\ell - \xi'_\ell = \lambda'_\ell \in {}^{B}\mathfrak{S}_{GNI}^{ext}$, in the restriction from ${}^{B}\mathcal{H}_{GNI}^{ext}$ to $\mathcal{H}_{GNI}$, implies that the closure of the local states associated to the field algebra $\mathfrak{S}_{GNI}$ intrinsic to the model does not allow the introduction of the operator : $\exp\{2i\sqrt{\pi}(\theta_\ell - \xi'_\ell)\}$:, which is charged under $\mathcal{Q}_{\tilde{\lambda}'}$. This implies that the field operator (see Eq.(2.47))

$$S_{Sch} \doteq: e^{+2ig\lambda'_\ell}\psi_\ell : \ |_{a=2} \equiv: e^{-2i\sqrt{\pi}(\theta_\ell - \xi'_\ell)} : \psi_{\ell GNI} =: \psi'_{\ell GI} \ e^{2i\sqrt{\pi}\theta_r} : \ , \tag{3.29}$$

cannot be defined on $\mathcal{H}_{GNI}$ and does not belong to the field algebra $\mathfrak{S}_{GNI}$.

The enlargement of the Boson field algebra $\mathfrak{S}_{GNI}^{B}$ performed in Ref.[9], with the introduction of the dual WZ field

$$\tilde{\theta} = \tilde{\xi}' - \frac{g}{\sqrt{\pi}}\left(\frac{1}{a-1}\chi + \lambda\right) , \ \forall\, a > 1 \ , \tag{3.30}$$

implies that

$$\theta_\ell = \xi'_\ell \, , \tag{3.31}$$

and thus $\lambda'_\ell \equiv 0$. In this case the field operator (3.30) reduces to

$$\psi_{\ell_{GNI}} =: \psi'_{\ell_{GI}} \, e^{2\,i\,\sqrt{\pi}\,\theta_r} \; : \; . \tag{3.32}$$

For the moment consider the following commutation relations

$$[\phi_\ell(x) - \theta(x) \, , \, L_{GI}(y)] = 0 \, , \tag{3.33a}$$

$$[\phi_\ell(x) - \theta_r(x) \, , \, L_{GI}(y)] = [\theta_\ell(x) \, , \, L_{GI}(y)] = \left(\frac{a-2}{a-1}\right)\Delta(x^+ - y^+; 0) \, , \tag{3.33b}$$

where L_{GI} is the potential for the longitudinal current L^μ_{GI}. From these commutation relations we see that for values of the regularization parameter in the range $a > 1$, the field combination $\phi_\ell - \theta$ satisfies the subsidiary condition (3.21). However, for the special value $a = 2$, there is a broader class of operators belonging to ${}^B\mathfrak{S}^{ext}_{GI}$ that satisfy the subsidiary condition and include the field combination $\phi_\ell - \theta_r$ and the field $\theta_\ell = \xi'_\ell$, which in this case decouples from the longitudinal current (3.14b) but not from the Lagrangian density, as occurs in the VSM.

However, the use of the external Bose algebra ${}^B\mathfrak{S}^{ext}_{GNI}$ makes the problem more dramatic since the trivialization of the charge $\mathcal{Q}_{\tilde\theta}$, associated with the WZ field component θ_r, implies that the Wick exponential $: \exp\{2\,i\,\sqrt{\pi}\,\theta_r\} :$ cannot be defined either on the Hilbert space $\mathcal{H}_{GNI}$ or on $\mathcal{H}_{GI}$. In this case the definition of the operator $\psi'_{\ell_{GI}}$ is meaningless as well[17].

At a glance, the isomorphism between GNI and GI Wightman functions points towards on behalf of gauge invariance such that the rhs of the identity (3.26) exhausts all possible

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[17] *Since for $a = 2$ the field combination $(\phi_\ell - \theta_r) \in {}^B\mathfrak{S}^{ext}$ satisfies the subsidiary condition, the authors of Ref.[9] chose the field operator ${}^{\theta_r}\psi_\ell \notin {}^\theta\mathfrak{S}$ instead of ${}^\theta\psi_\ell \in \mathfrak{S}_{GI}$ to define the physical fermionic content of the model. The gauge transformed operator (2.50) is given by*

$$^\theta\Psi(x) = \left(\frac{\mu_o}{2\pi}\right)^{1/4} \; : \; e^{i\,\sqrt{\pi}\,\gamma^5\,\chi_1(x)} \; : \; e^{i\,\sqrt{\pi}\,[\gamma^5\,{}^\theta\tilde{L}(x) + {}^\theta L(x)]} \; : \, ,$$

observable content of the theory. As a matter of fact, the isomorphism established by Eqs.(3.26) has an algebraic interpretation[18] and in order to be correctly understood must be seen on the basis of the Wightman reconstruction theorem in the sense that the set of Wightman functions $\langle \wp \{ \bar{\psi}_{GNI}, \psi_{GNI}, \mathcal{A}^{\nu}_{GNI} \} \rangle$ $(\langle \wp \{ \bar{\psi}_{GI}, \psi_{GI}, \mathcal{A}^{\nu}_{GI} \} \rangle)$ uniquely determine (up to isomorphisms) a representation of the field algebra $\Im_{GNI}$ $(\Im_{GI})$.

For the GNI and GI formulations we have the realization of the corresponding weak local Gauss's law

$$\langle \, \Phi_{\alpha} \, , \, (\, \mathcal{J}^{0}_{\alpha}(x) \, - \, \partial_1 \, \mathcal{F}^{10}(x) \,) \, \Psi_{\alpha} \, \rangle \; = \; \langle \, \Phi_{\alpha} \, , \, L^{0}_{\alpha}(x) \, \Psi_{\alpha} \, \rangle \; = \; 0 \qquad (3.34)$$

where α means that we are working in GI or GNI formulations. We can construct the local charges

$$\mathcal{Q}_{\alpha}[\Lambda] \; = \; \int_{z^o} \Lambda(z) \left(\mathcal{J}^{0}_{\alpha}(z) - \partial_1 \, \mathcal{F}^{10}(z) \right) dz^1 \; = \; \int_{z^0} \{ \, L^{0}_{\alpha}(z) \, \Lambda(z) - L^{1}_{\alpha}(z) \, \tilde{\Lambda}(z) \, \} \, dz^1 \; , \quad (3.35)$$

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and does not belong to $^{\theta}\Im$ either. However, the fact that the operator $^{\theta}\Psi$ of the GI formulation (or Ψ of the GNI formulation) shares the same features of the fermion operator of the covariant solution of the VSM [13] does not imply the equivalence between the two models. In order for this to be true, such an equivalence should be established between all Wightman functions of the two models and this is not the case, since neither $\Psi \notin \Im_{GNI}$ can be defined on the Hilbert space $\mathcal{H}_{GNI}$ of the GNI formulation nor $^{\theta}\Psi \notin \Im_{GI}$ can be defined in the Hilbert space $\mathcal{H}_{GI}$ of the GI formulation.

The Hilbert space $^{\theta}\mathcal{H} \equiv \mathcal{H}_{GI}$ can be factorized as a product

$$^{\theta}\mathcal{H} = \mathcal{H}_{\chi_1} \otimes \mathcal{H}_{\chi_2, \xi', \psi_o} \ .$$

The Hilbert space completion $\mathcal{H}_{\chi_2, \xi', \psi_o}$ of the local states of the GI formulation is isomorphic to the closure of the space of the GNI formulation $\mathcal{H}_{\chi_2, \lambda', \psi_o}$, and also cannot be further decomposed.

[18] *As stressed in Ref.[6b] within the path-integral formalism, this isomorphism only is implemented on full quantum level where the integration over $\mathcal{A}_{\mu}$ has also been performed. The isomorphism expressed by the identity (3.26) does not exist on the level where $\mathcal{A}_{\mu}$ is an external field.*

with $\Box \Lambda = 0$, and which are implementable on the Hilbert space $\mathcal{H}_\alpha$. The corresponding charge generators are obtained as weak limits of the above charges and the unitary operators

$$T_\alpha[\Lambda] = e^{-i\frac{\sqrt{\pi}}{g}Q_\alpha[\Lambda]} \tag{3.36}$$

implement the local gauge transformations. Since we are dealing with an anomalous theory, the field algebra $\mathfrak{S}_{GNI}$ ($\mathfrak{S}_{GI}$) is a singlet under local gauge transformations generated by the operator $T_{GNI}[\Lambda] \in \mathfrak{S}_{GNI}$ ($T_{GI}[\Lambda] \in \mathfrak{S}_{GI}$). In view of Eqs.(3.17-18), we can consider the field algebra $\mathfrak{S}_{GI}$ as a field subalgebra of $\mathfrak{S}_{GNI} \supset \mathfrak{S}_{GI}$ such that the operator $T_{GNI}[\Lambda]$ implements the local c-number extended gauge transformations:

$$T_{GNI}[\Lambda] \, \psi_{GI}(x) \, T_{GNI}^{-1}[\Lambda] = e^{2i\sqrt{\pi}\Lambda(x)} \, \psi_{GI}(x) \;, \tag{3.37a}$$

$$T_{GNI}[\Lambda] \, \mathcal{A}_{GI}^\mu(x) \, T_{GNI}^{-1}[\Lambda] = \mathcal{A}_{GI}^\mu(x) + \frac{\sqrt{\pi}}{g} \partial^\mu \Lambda(x) \;, \tag{3.37b}$$

$$T_{GNI}[\Lambda] \, \theta(x) \, T_{GNI}^{-1}[\Lambda] = \theta(x) - \Lambda(x) \;, \tag{3.37c}$$

$$T_{GNI}[\Lambda] \, \psi_{GNI}(x) \, T_{GNI}^{-1}[\Lambda] = \psi_{GNI}(x) \;, \tag{3.37d}$$

$$T_{GNI}[\Lambda] \, \mathcal{A}_{GNI}^\mu(x) \, T_{GNI}^{-1}[\Lambda] = \mathcal{A}_{GNI}^\mu(x) \;, \tag{3.37e}$$

and similarly for the operator $T_{GI}[\Lambda]$ (with $\Lambda \to -\Lambda$ in the analog of the transformation (3.37c)), under which the field algebra $\mathfrak{S}_{GI}$ is a singlet.

From Eq. (3.27c) we see that the generator that is implementable in both GNI and GI Hilbert spaces is obtained as weak limit of the local charge operator constructed from the WZ current $\mathcal{J}_{WZ}^\mu$ which, in view of Eq.(3.14), can be written as

$$\mathcal{Q}_{WZ} = \mathcal{Q}_{GI} - \mathcal{Q}_{GNI} \;. \tag{3.38}$$

In the "embedded" formulation of the anomalous model the local gauge invariance is recovered at the quantum level, meaning that there exists an unitary operator

$$T_{WZ}[\Lambda] = e^{-i\frac{\sqrt{\pi}}{g}Q_{WZ}[\Lambda]} \tag{3.39}$$

which implements the extended local gauge transformations

$$T_{WZ}[\Lambda] \, \psi_\alpha(x) \, T_{WZ}^{-1}[\Lambda] = e^{\pm 2i\sqrt{\pi}\Lambda(x)} \, \psi_\alpha(x) \;, \tag{3.40a}$$

33

$$T_{WZ}\left[\Lambda\right]\mathcal{A}_\alpha^\mu(x)\,T_{WZ}^{-1}[\Lambda] = \mathcal{A}_\alpha^\mu(x) \pm \frac{\sqrt{\pi}}{g}\,\partial^\mu\Lambda(x)\;, \tag{3.40b}$$

$$T_{WZ}\left[\Lambda\right]\theta(x)\,T_{WZ}^{-1}[\Lambda] = \theta(x) + 2\,\Lambda(x)\;, \tag{3.40c}$$

with α standing for GNI and GI field operators, respectively. The local gauge invariance manifests itself by the existence of the generator $\mathcal{Q}_{WZ}$ under whose action the isomorphic intrinsic local field algebras $\Im_{GNI}$ and $\Im_{GI}$ are not singlets.

Contrary to what happens in the VSM [15], the CSM is not a confining gauge theory and, due to the absence of spurious vacuum raising operators belonging to the intrinsic field algebra, the model does not exhibit a topological vacuum structure.

IV. CONCLUSIONS

A careful analysis of some basic structural properties of the anomalous CSM in the isomorphic GNI and GI operator formulations was performed. In our treatment we only use the intrinsic field algebra generated from the irreducible set of field operators of the theory in order to construct the Hilbert space associated with the Wightman functions that define the model. By relaxing this careful control on the construction of the Hilbert space of the theory one may carry along a lot of superfluous states and some misleading conclusions may arise.

A peculiar feature of the CSM which differs from the vector case is the fact that the cluster decomposition property is not violated for Wightman functions that are representations of the field algebra $\Im$ generated from the irreducible set of field operators $\{\,\bar\psi,\psi,\mathcal{A}_\mu\,\}$. The need of the Θ-vacuum parametrization and the equivalence of the CSM defined for $a = 2$ and the VSM, only emerge due to an improper factorization of the closure of the Hilbert space $\mathcal{H}$ which defines the representation of the intrinsic field algebra $\Im$. The main results of Refs.[9,10] cannot be regarded as structural properties of the CSM since they are consequences of the use of a redundant field algebra belonging to the external algebra $\Im^B$ rather than on the use of the intrinsic field algebra $\Im$ which defines the model.

34

References

1 - R. Jackiw and R. Rajaraman, *Phys. Rev. Lett.* **B 54** (1985), 1219.

2 - In addition the references quoted here, see also

- R. Rajaraman, *Phys. Lett.* **B 514** (1985), 305;

- J. Lott and R. Rajaraman, *Phys. Lett.* **B 165** (1985), 321;

- A. Niemi and G. Semenoff, *Phys. Lett.* **B 175** (1986), 439;

- M. Chanowitz, *Phys. Lett.* **B 171** (1986), 280;

- N. K. Falck and G.Kramer, *Ann. Phys. (N.Y.)* **176** (1987), 330;

- C. Aragão de Carvalho, K. D. Rothe, C. A. Linhares and H. J. Rothe, *Phys. Lett.* **B 194**
  (1987), 539.

3 - L. D. Faddeev and L. Shatashvili, *Phys. Lett.* **B 176** (1986), 225;

- L. D. Faddeev, *Phys. Lett.* **B 145** (1984), 81.

4 - J. Wess and B. Zumino, *Phys. Lett.* **B 37** (1971), 95.

5 - O. Babelon,F. A. Shaposnik and C. M. Viallet, *Phys. Lett.* **B 177** (1986), 385;

- K. Harada and I. Tsutsui, *Phys. Lett.* **B 183** (1987), 311.

6 - H. O. Girotti and K. D. Rothe, *Int. J. Mod. Phys.* **A 12** (1989), 3041;

- E. Abdalla, M. C. Abdalla and K. D. Rothe, "Non-Perturbative Methods in 2 Dimensional Quantum Field Theory", World Scientific, 1991.

7 - L. V. Belvedere, K. D. Rothe, *Mod. Phys. Lett.* **A 10**, (1995), 207.

8 - D. Boyanovsky, I. Schmidt and M. F. L. Golterman, *Ann. Phys. (N.Y.)* **185** (1988), 111.

9 - M. Carena and C. E. M. Wagner, *Int. Jour. Mod. Phys*, **A 6** (1991), 243.

10- K. Shizuya, *Phys. Lett.* **B 213** (1988), 298;

   - S. Miyake and K. Shizuya, *Phys. Rev.* **D 36** (1987), 3791;

   - T. Berger, N. K. Falck and G. Kramer, *Int. Jour. Mod. Phys.* **A 4** (1989) 427.

11- G. Morchio, D. Pierotti and F. Strocchi, *Ann. Phys. (N.Y.)* **188** (1988), 217;

   - F. Strocchi, "Selected Topics on the General Properties of Quantum Field Theory", Lecture Notes in Physics, vol.51, World Scientific Publishing, 1993; see also

   - A. Z. Capri and R. Ferrari, *Nuovo Cimento* **A 62** (1981), 273.

12- H. O. Girotti, H. J. Rothe and K. D. Rothe, *Phys. Rev.* **D 34** (1986), 592.

13- J. H. Lowenstein and J. Swieca, *Ann. Phys. (N.Y.)* **68** (1971), 172;

   - L. V. Belvedere, J. A. Swieca, K. D. Rothe and B. Schroer, *Nucl. Phys.* **B 153** (1979), 112;

   - L. V. Belvedere, *Nucl. Phys.* **B 276** (1986), 197.

14- M. B. Halpern, *Phys. Rev.* **D 13** (1976), 337.

15- K. D. Rothe and J. Swieca, *Phys. Rev.* **D 15** (1977), 541.

16- R. F. Streater and A. S. Wightman, "PCT, Spin, Statistics and All That" W. A. Benjamin, Inc, New York, 1964.

   - R. Jost, "Properties of Wightman Functions" in Lectures on Field Theory and the Many-Body Problem, (E. R. Caianiello Ed.), Academic Press, New York, 1961;

- R. Haag, "it Local Quantum Physics, Fields, Particles, Algebras", Texts and Monographs in Physics, Springer-Verlag, 1992.

17 - D. Pierotti, *Lett. Math. Phys.* **15** (1988), 219.

18 - H. O. Girotti, H. J. Rothe and K. D. Rothe, *Phys. Rev.* **D 33** (1986), 514;

- H. O. Girotti, H. J. Rothe and K. D. Rothe, *Phys. Rev.* **D 34** (1986) 592;

- N. K. Falck and G.Kramer, *Ann. Phys. (N.Y.)* **176** (1987), 330.

19 - Wightman, A. S., in M. Levy (ed.), "Cargèse Lectures in Theoretical Physics" 1964 (1967).